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Approximate Vibration Reanalysis of Structures

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The combined approximations method is developed for vibration reanalysis of structures. In the solution process, the terms of local (series) approximations are used as basis vectors in global (reduced basis) approximations. The solution steps are straightforward, and the method can be readily used with a general finite element system. The computational effort needed by the method is usually much smaller than the effort needed for complete vibration analysis. The efficiency of the calculations and the accuracy of the results can be controlled by the amount of information considered. Efficient solutions are obtained by low-order approximations and accurate results can be achieved by considering higher-order terms. Numerical examples show that accurate results are achieved efficiently for very large changes in the design.

Nomenclature

$\boldsymbol{B}$	=	matrix defined by Eq. (21)
$\boldsymbol{C}$	=	matrix defined by Eq. (16)
E	=	modulus of elasticity
I	=	moment of inertia
$\boldsymbol{I}$	=	identity matrix
$\boldsymbol{K}$	=	stiffness matrix
$\boldsymbol{K}_R$	=	reduced stiffness matrix
L	=	length
$\boldsymbol{M}$	=	mass matrix
$\boldsymbol{M}_R$	=	reduced mass matrix
$\boldsymbol{R}$	=	right-hand-side vector
$\boldsymbol{r}$	=	displacement vector
$\boldsymbol{r}_B$	=	matrix of basis vectors
s	=	number of basis vectors
$\boldsymbol{U}$	=	upper triangular matrix
$\boldsymbol{y}$	=	vector of coefficients
α	=	coefficient defined by Eq. (35)
β	=	angle between vectors
$\Delta\boldsymbol{K}$	=	change in the stiffness matrix
$\Delta\boldsymbol{M}$	=	change in the mass matrix
δ	=	Kronecker delta
λ	=	eigenvalue
ω	=	circular frequency

I. Introduction

REPEATED analysis or reanalysis is needed in various problems of structural analysis, design, and optimization. In general, the structural response cannot be expressed explicitly in terms of the design variables, and structural analysis involves the solution of a set of simultaneous equations. Reanalysis methods are intended to analyze efficiently structures that are modified due to changes in the design. The object is to evaluate the structural response (for example, displacements, forces, and stresses) for such changes without solving the complete set of modified equations. The solution procedures usually use the original response of the structure.

Several comprehensive reviews on reanalysis methods have been published in the past.[1−3] For approximate reanalysis, the accuracy of the results (the quality of the approximations) and the efficiency of the calculations (the computational effort involved) are usually two conflicting factors. That is, better approximations are often achieved at the expense of more computational effort. The common approximations can be divided into the following classes[4]:

1) The first class comprises local approximations (also called single-point approximations), such as the Taylor series expansion or the binomial series expansion about a given design point. Local approximations are based on information calculated at a single point. These methods are very efficient, but they are effective only for small changes in the design variables. For large changes in the design, the accuracy of the approximations often deteriorates and the results may become meaningless.

2) The second class comprises global approximations (also called multipoint approximations), such as polynomial fitting, response surface, or reduced basis methods.[5−7] These approximations are obtained by analyzing the structure at a number of design points, and they are valid for the whole design space (or, at least, large regions of it). Global approximations may require much computational effort, particularly in problems with large numbers of design variables.

3) In this study, a third class of approximations, called combined approximations, is considered. In this approach we attempt to give global qualities to local approximations. This can be achieved by considering the terms of local (series) approximations as basis vectors in global (reduced basis) approximations. It was found in previous studies[8,9] that this choice of basis vectors provides accurate results for linear static problems.

The purpose of dynamic analysis is to determine internal forces, stresses, and displacements under application of dynamic (time-varying) loads. The problem of vibration analysis (or eigenproblem) consists of determining the conditions under which the equilibrium conditions are satisfied.[10,11] In vibration mode superposition analysis, the main computational effort is spent in the solution of the eigenprolem, which requires considerably more effort than a static analysis. Because exact solution of the eigenproblem can be prohibitively expensive, approximate solution techniques have been developed, primarily to calculate the lowest eigenvalues and corresponding eigenvectors.

Most reanalysis methods are suitable only for linear static analysis. Several studies have been published on eigenvalue reanalysis.[12−14] Vibration reanalysis, where the object is to calculate both the modified eigenvectors and eigenvalues, has been studied by several authors.[15,16]

In this study, the combined approximations (CA) method is developed for vibration reanalysis. In Sec. II, the problem is introduced; the analysis equations are presented and the reanalysis problem is formulated. In Sec. III, reanalysis by the CA method is developed; the reanalysis equations are presented and the solution procedure is described. Numerical examples are demonstrated in Sec. IV, and the conclusions are drawn in Sec. V.

II. Problem Formulation

A. Vibration Analysis

The problem of vibration analysis is to find the free-vibration frequencies and the mode shapes of the vibrating system. The

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solution of the eigenproblem for large structures is often the most costly phase of a dynamic response analysis, and calculation of the eigenvalues and eigenvectors requires much computational effort.

Consider the generalized eigenproblem

$$\boldsymbol{K}\boldsymbol{r} = \omega^2 \boldsymbol{M}\boldsymbol{r} = \lambda \boldsymbol{M}\boldsymbol{r} \tag{1}$$

where $\lambda = \omega^2$ are the eigenvalues indicating the square of the free-vibration frequencies and $\boldsymbol{r}$ are the corresponding eigenvectors, or mode shapes, of the vibrating system. For a system having n degrees of freedom, the frequency vector $\boldsymbol{\omega}^T = \{\omega_1, \omega_2, \omega_3, \ldots, \omega_N\}$ represents the frequencies of the n possible modes of vibration

A basic question in considering an effective solution method is whether we should first solve for the eigenvalue and then calculate the eigenvector, or vice versa, or whether it is more economical to solve for both simultaneously. The answer to this question depends on various properties of the problem under consideration. The common solution methods can be subdivided into the following groups, corresponding to the basic properties used in the solution process[11]: 1) vector iteration methods (for example, inverse iteration), 2) transformation methods (for example, Jacobi iteration), 3) polynomial iteration methods, and 4) methods that employ the sequence property of the characteristic polynomials. There are many variants of these procedures. The Lanczos method and the subspace iteration method use a combination of the properties used in the preceding methods.

Many of the vibration solution techniques are based on inverse iteration.[10,11] To calculate the first-mode shape, Eq. (1) can be rewritten in an iterative form. Because only the shape is needed, the frequency is dropped to obtain

$$\boldsymbol{K}\bar{\boldsymbol{r}}_k = \boldsymbol{M}\boldsymbol{r}_{k-1} \tag{2}$$

where k denotes the iteration number, $\boldsymbol{r}_{k-1}$ is the displacement vector in the previous iteration, and $\bar{\boldsymbol{r}}_k$ is the resulting improved shape. To initiate the iteration procedure, an initial displacement vector $\boldsymbol{r}_0$ is assumed that is a reasonable estimate of this shape. The improved iteration vector is then obtained by normalizing the shape $\bar{\boldsymbol{r}}_k$ by dividing it by a reference ref$(\bar{\boldsymbol{r}}_k)$ to obtain

$$\boldsymbol{r}_k = \bar{\boldsymbol{r}}_k / \text{ref}(\bar{\boldsymbol{r}}_k) \tag{3}$$

In general, the vector is normalized with respect to its largest element. By repeating the process sufficiently, we can improve the mode-shape approximation to any desired level of accuracy. After s iteration cycles, the frequency can be obtained by

$$\omega_1^2 = \lambda_1 = \frac{\max(\boldsymbol{r}_{s-1})}{\max(\bar{\boldsymbol{r}}_s)} = \frac{1}{\max(\bar{\boldsymbol{r}}_s)} \tag{4}$$

Inverse iteration can be used to evaluate higher-order modes as well, by assuming shapes that contain no lower-mode components.

B. Vibration Reanalysis

Consider an initial design with stiffness matrix $\boldsymbol{K}_0$ and mass matrix $\boldsymbol{M}_0$. The corresponding eigenvectors $\boldsymbol{r}_0$ and eigenvalues λ_0 are calculated by solving the set of initial analysis equations

$$\boldsymbol{K}_0\boldsymbol{r}_0 = \lambda_0 \boldsymbol{M}_0\boldsymbol{r}_0 \tag{5}$$

The symmetric positive-definite stiffness matrix $\boldsymbol{K}_0$ is usually given from the initial analysis in the decomposed form

$$\boldsymbol{K}_0 = \boldsymbol{U}_0^T \boldsymbol{U}_0 \tag{6}$$

where $\boldsymbol{U}_0$ is an upper triangular matrix.

Assume a change in the design and corresponding changes in the stiffness and mass matrices such that the modified matrices are expressed as

$$\boldsymbol{K} = \boldsymbol{K}_0 + \Delta\boldsymbol{K}, \qquad \boldsymbol{M} = \boldsymbol{M}_0 + \Delta\boldsymbol{M} \tag{7}$$

Substitution of Eq. (7) into Eq. (1) gives the modified analysis equations

$$(\boldsymbol{K}_0 + \Delta\boldsymbol{K})\boldsymbol{r} = \lambda \boldsymbol{M}\boldsymbol{r} \tag{8}$$

The reanalysis problem under consideration can be stated as follows. Given the initial values $\boldsymbol{K}_0$ and $\boldsymbol{M}_0$, the initial eigenvectors $\boldsymbol{r}_0$ and eigenvalues λ_0 are first calculated by solving Eq. (5). The object is to evaluate the modified eigenvectors $\boldsymbol{r}$ and eigenvalues λ due to various changes in the design, such that the modified analysis equations (8) are satisfied.

In eigenproblems, a distinction is made between the following two types of reanalysis:

1) In eigenvalue reanalysis, the object is to calculate the modified eigenvalues due to changes in the stiffness and mass matrices. It is often assumed that the mode shapes change only insignificantly due to changes in the design. Several studies are devoted to approximations just for frequency problems.[12–14] Eigenvalue reanalysis is not considered in this study.

2) In vibration reanalysis,[15,16] the object is to calculate both the modified eigenvectors and eigenvalues due to changes in the design. This type of problem, which is more difficult to solve than the eigenvalue reanalysis problem, is considered in this study.

The reanalysis problem under consideration could be solved by the inverse iteration method [Eq. (2)]. However, in this case it would be necessary to factorize the modified stiffness matrix for each change in the design. In the approach presented in this study, we use the decomposed form of the initial stiffness matrix [Eq. (6)], which is given from the initial analysis.

III. Reanalysis by the CA Method

A. Reduced Modified Equations

In a typical reduced-basis method we assume that the mode shape $\boldsymbol{r}$ of a new design can be approximated by a linear combination of preselected s linearly independent basis vectors, also called global approximation vectors, $\boldsymbol{r}_1, \boldsymbol{r}_2, \ldots, \boldsymbol{r}_s$:

$$\boldsymbol{r} = y_1\boldsymbol{r}_1 + y_2\boldsymbol{r}_2 + \cdots + y_s\boldsymbol{r}_s = \boldsymbol{r}_B\boldsymbol{y} \tag{9}$$

where we assume that s is much smaller than the number of degrees of freedom n, $\boldsymbol{r}_B$ is the $n \times s$ matrix of the basis vectors, and $\boldsymbol{y}$ is to be determined:

$$\boldsymbol{r}_B = [\boldsymbol{r}_1, \boldsymbol{r}_2, \ldots, \boldsymbol{r}_s] \tag{10}$$

$$\boldsymbol{y}^T = \{y_1, y_2, \ldots, y_s\} \tag{11}$$

The justification for using this approach is that the large number of degrees of freedom describing the response of the system is often dictated by such considerations as complex topology or numerous changes in the system properties, rather than by the complexity of the response. The power of the reduced basis method derives from the fact that, for many systems of practical interest, the transformation of Eq. (9) can provide highly accurate approximations of $\boldsymbol{r}$, even when s is much smaller than n.

The modified analysis equations are now approximated by a smaller system of equations in the new unknowns $\boldsymbol{y}$. Substituting Eq. (9) into Eq. (1) and premultiplying the resultant equation by $\boldsymbol{r}_B^T$ gives the $s \times s$ system

$$\boldsymbol{r}_B^T\boldsymbol{K}\boldsymbol{r}_B\boldsymbol{y} = \lambda\boldsymbol{r}_B^T\boldsymbol{M}\boldsymbol{r}_B\boldsymbol{y} \tag{12}$$

Introducing the notation

$$\boldsymbol{K}_R = \boldsymbol{r}_B^T\boldsymbol{K}\boldsymbol{r}_B, \qquad \boldsymbol{M}_R = \boldsymbol{r}_B^T\boldsymbol{M}\boldsymbol{r}_B \tag{13}$$

and substituting Eq. (13) into Eq. (12), we obtain

$$\boldsymbol{K}_R\boldsymbol{y} = \lambda\boldsymbol{M}_R\boldsymbol{y} \tag{14}$$

The $s \times s$ matrix $\boldsymbol{K}_R$ is full but is symmetric and much smaller in size than the $n \times n$ matrix $\boldsymbol{K}$ of the original system. That is, rather than computing the exact solution by solving the large $n \times n$ system in Eq. (1), we first solve the smaller $s \times s$ system in Eq. (14) for $\boldsymbol{y}$ and then evaluate the approximate mode shape $\boldsymbol{r}$ for the computed $\boldsymbol{y}$ by Eq. (9).

B. Determining the Basis Vectors

The effectiveness of the reduced-basis method depends, to a great extent, on the appropriate choice of the basis vectors that span the reduced basis subspace. Proper selection of the basis vectors is perhaps the most important factor effecting the successful application of the method. It is shown in this study that the basis vectors determined by the CA method provide accurate results with a small computational effort for very large changes in the design.

The basis vectors could be determined by the inverse iteration method [Eq. (2)]. However, it has been noted that each change in the design requires factorization $\boldsymbol{K} = \boldsymbol{U}^T \boldsymbol{U}$ of the modified stiffness matrix, which involves much computational effort. To achieve an efficient solution procedure we drop the frequency and rearrange Eq. (8) to obtain the recurrence relation

$$\boldsymbol{K}_0 \bar{\boldsymbol{r}}_k = (-\Delta \boldsymbol{K} + \boldsymbol{M}) \boldsymbol{r}_{k-1} \tag{15}$$

Denoting

$$\boldsymbol{C} = \boldsymbol{K}_0^{-1} (\Delta \boldsymbol{K} - \boldsymbol{M}) \tag{16}$$

we calculate the basis vectors successively by

$$\bar{\boldsymbol{r}}_k = -\boldsymbol{C} \boldsymbol{r}_{k-1}, \qquad k = 2, \ldots, s \tag{17}$$

where the first basis vector is the initial mode shape $\boldsymbol{r}_0$:

$$\boldsymbol{r}_1 = \boldsymbol{r}_0 \tag{18}$$

Substituting Eq. (6) into Eq. (15), we obtain the recurrence relation

$$\boldsymbol{U}_0^T \boldsymbol{U}_0 \bar{\boldsymbol{r}}_k = (-\Delta \boldsymbol{K} + \boldsymbol{M}) \boldsymbol{r}_{k-1} \tag{19}$$

It can be observed that calculation of each basis vector by Eq. (19) involves only forward and backward substitutions because $\boldsymbol{U}_0$ is given from the initial analysis.

The basis vector $\bar{\boldsymbol{r}}_k$ can be normalized by Eq. (3):

$$\boldsymbol{r}_k = \bar{\boldsymbol{r}}_k / \mathrm{ref}(\bar{\boldsymbol{r}}_k) \tag{20}$$

Alternatively, the basis vectors can be determined as follows. Premultiply Eq. (8) by $\boldsymbol{K}_0^{-1}$ and denote

$$\boldsymbol{B} = \boldsymbol{K}_0^{-1} \Delta \boldsymbol{K} \tag{21}$$

$$\boldsymbol{R} = \lambda \boldsymbol{M} \boldsymbol{r} \tag{22}$$

$$\boldsymbol{r}_1 = \boldsymbol{K}_0^{-1} \boldsymbol{R} \tag{23}$$

The resulting equation is

$$(\boldsymbol{I} + \boldsymbol{B}) \boldsymbol{r} = \boldsymbol{r}_1 \tag{24}$$

Premultiplying Eq. (24) by $(\boldsymbol{I} + \boldsymbol{B})^{-1}$ and expanding,

$$(\boldsymbol{I} + \boldsymbol{B})^{-1} \cong \boldsymbol{I} - \boldsymbol{B} + \boldsymbol{B}^2 - \cdots \tag{25}$$

gives the series

$$\boldsymbol{r} = (\boldsymbol{I} - \boldsymbol{B} + \boldsymbol{B}^2 - \cdots) \boldsymbol{r}_1 \tag{26}$$

The terms of the series of Eq. (26) could be assumed as basis vectors

$$\boldsymbol{r}_1 = K_0^{-1} \boldsymbol{R} \tag{27}$$

$$\boldsymbol{r}_k = -\boldsymbol{B} \boldsymbol{r}_{k-1}, \qquad k = 2, \ldots, s \tag{28}$$

However, because $\boldsymbol{R}$ is not known [Eq. (22)], it is convenient to consider $\boldsymbol{R}_0 = \lambda_0 \boldsymbol{M} \boldsymbol{r}_0$ instead of $\boldsymbol{R} = \lambda \boldsymbol{M} \boldsymbol{r}$ in the calculation of $\boldsymbol{r}_1$. The advantage is that the elements of $\boldsymbol{R}_0$ are already known. It is found that, for this selection of $\boldsymbol{r}_1$, accurate results are achieved with a small number of basis vectors. Thus, the first basis vector is given by

$$\boldsymbol{r}_1 = \boldsymbol{K}_0^{-1} \boldsymbol{R}_0 \tag{29}$$

The additional vectors are calculated successively by Eq. (28) and normalized by Eq. (20):

$$\bar{\boldsymbol{r}}_k = -\boldsymbol{B} \boldsymbol{r}_{k-1}, \qquad k = 2, \ldots, s \tag{30}$$

$$\boldsymbol{r}_k = \bar{\boldsymbol{r}}_k / \mathrm{ref}(\bar{\boldsymbol{r}}_k) \tag{31}$$

It can be shown that for $\boldsymbol{M} = \boldsymbol{M}_0$ both the methods of Eq. (17) and of Eq. (30) provide identical results.

The procedure presented can be used to determine the basis vectors for various mode shapes. For the mth mode shape $\boldsymbol{r}m$, we use the corresponding initial value $\boldsymbol{r}m_0$ as the first basis vector. It is shown by numerical examples in Sec. IV that high accuracy is achieved for the first mode shape, with a small number of basis vectors, for very large changes in the design. Less accurate results might be obtained for the higher mode shapes. The accuracy can be improved significantly with a small computational effort by using vector deflation.[11] The basis of vector deflation is that to converge to a required eigenvector, the iteration vector must not be orthogonal to it. If the iteration vector is orthgonalized to the eigenvectors already calculated, we eliminate the possibility that the iteration converges to any one of them, and convergence occurs to another eigenvector.

Assume that we have calculated the first m eigenvectors, $\boldsymbol{r}1, \boldsymbol{r}2, \ldots, \boldsymbol{r}m$, and that we want to $\boldsymbol{M}$ orthogonalize $\bar{\boldsymbol{r}}(m+1)$ to these eigenvectors. When the Gram–Schmidt orthogonalization is considered, a vector $\boldsymbol{r}(m+1)$ that is $\boldsymbol{M}$ orthogonal to the eigenvectors $\boldsymbol{r}i$ $(i = 1, \ldots, m)$ is calculated using

$$\boldsymbol{r}(m+1) = \bar{\boldsymbol{r}}(m+1) - \sum_{i=1}^{m} \alpha_i \boldsymbol{r}i \tag{32}$$

The coefficients α_i are obtained from the conditions

$$\boldsymbol{r}i^T \boldsymbol{M} \boldsymbol{r}(m+1) = 0, \qquad i = 1, \ldots, m \tag{33}$$

$$\boldsymbol{r}i^T \boldsymbol{M} \boldsymbol{r}j = \delta_{ij} \tag{34}$$

where δ_{ij} is the Kronecker delta, for which $\delta_{ij} = 0$ $(i \neq j)$ and 1. Premultiplying both sides of Eq. (32) by $\boldsymbol{r}i^T \boldsymbol{M}$, we obtain

$$\alpha_i = \boldsymbol{r}i^T \boldsymbol{M} \bar{\boldsymbol{r}}(m+1) \tag{35}$$

In the iteration, we would now use $\boldsymbol{r}(m+1)$ as the starting iteration vector instead of $\bar{\boldsymbol{r}}(m+1)$, and convergence occurs to the $(m+1)$th eigenvector.

Note that for changes in a single element or a small number of elements of the structure, the basis vectors may become linearly dependent. This means that consideration of additional vectors provides no useful information. Two basis vectors $\boldsymbol{r}_k$ and $\boldsymbol{C}\boldsymbol{r}_k$, determined by Eqs. (17) and (20), are close to being linearly dependent if

$$\cos \beta_{k,k+1} = \frac{\left(\boldsymbol{r}_k^T \boldsymbol{C} \boldsymbol{r}_k\right)}{(|\boldsymbol{r}_k||\boldsymbol{C}\boldsymbol{r}_k|)} \approx 1 \tag{36}$$

where $\beta_{k,k+1}$ is the angle between the two vectors. Various numerical examples show that the basis vectors determined by the CA method satisfy the condition of Eq. (36) as the basis vectors index k is increased, even for very large changes in the design.

C. Solution Procedure

To clarify the presentation, the solution procedure is presented for a single mode shape. Considerations related to other mode shapes are introduced when necessary. Given the initial values $\boldsymbol{K}_0$, $\boldsymbol{M}_0$, $\boldsymbol{r}_0$, and λ_0 from vibration analysis of the initial structure, the reanalysis procedure to evaluate the modified eigenvector $\boldsymbol{r}$ and eigenvalue λ due to a change in the design involves the following steps:

1) Calculate the basis vectors [Eqs. (17), (18), and (20)]

$$\boldsymbol{r}_1 = \boldsymbol{r}_0 \tag{37}$$

$$\bar{\boldsymbol{r}}_k = -\boldsymbol{C} \boldsymbol{r}_{k-1}, \qquad k = 2, \ldots, s \tag{38}$$

$$r_k = \bar{\boldsymbol{r}}_k / \mathrm{ref}(\bar{\boldsymbol{r}}_k) \tag{39}$$

Table 1 Mode shape 1–4 mass changes in a cantilever beam

	Value for mode:							
Parameter	1	1	2	2	3	3	4	4
Case	$r1(2)$	$r1$(exact)	$r2(2)$	$r2$(exact)	$r3(2)$	$r3$(exact)	$r4(2)$	$r4$(exact)
Shape	1.0000	1.0000	1.0000	1.0000	0.7413	0.7411	0.5584	0.5580
	0.7664	0.7664	0.0184	0.0185	−0.6118	−0.6115	−0.9817	−0.9809
	0.5401	0.5401	−0.7142	−0.7145	−0.7657	−0.7663	0.1348	0.1350
	0.3331	0.3331	−0.9820	−0.9822	0.2295	0.2309	1.0000	1.0000
	0.1615	0.1615	−0.7574	−0.7571	1.0000	1.0000	−0.4690	−0.4711
	0.0438	0.0438	−0.2793	−0.2789	0.6104	0.6091	−0.9876	−0.9865
λ	0.0025	0.0025	0.0990	0.0990	0.7743	0.7743	2.9352	2.9352

Table 2 Mode shape 1–3 stiffness changes in all stories, eight-story frame

	Value for mode:					
Parameter	1	1	2	2	3	3
Case	$r1(2)$	$r1$(exact)	$r2(2)$	$r2$(exact)	$r3(2)$	$r3$(exact)
Shape	1	1	1	1	1	1
	0.97	0.97	0.77	0.77	0.44	0.44
	0.91	0.91	0.36	0.36	−0.37	−0.37
	0.82	0.82	−0.12	−0.12	−0.82	−0.82
	0.70	0.70	−0.57	−0.57	−0.66	−0.64
	0.54	0.54	−0.84	−0.84	0.02	0.03
	0.37	0.37	−0.84	−0.84	0.69	0.68
	0.19	0.19	−0.52	−0.52	0.64	0.64
λ	18.0	18.0	138.5	138.5	337.4	337.3

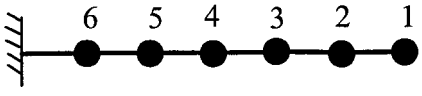


Fig. 1 Cantilever beam.

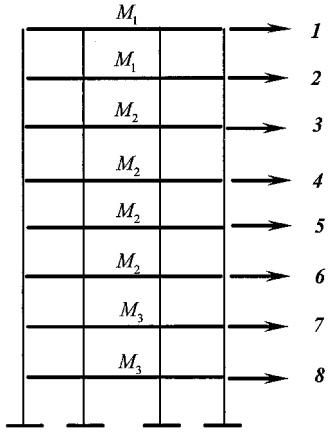


Fig. 2 Frame, eight stories.

where $\boldsymbol{C}$ is defined as [Eq. (16)]

$$\boldsymbol{C} = \boldsymbol{K}_0^{-1}(\Delta\boldsymbol{K} - \boldsymbol{M}) \tag{40}$$

It was noted that, if two basis vectors are close to being linearly dependent [Eq. (36)], consideration of additional vectors is not necessary. The number of basis vectors s required to obtain accurate results is usually much smaller than the number of degrees of freedom.

If we have calculated the first m eigenvectors, $\boldsymbol{r}1, \boldsymbol{r}2, \ldots, \boldsymbol{r}m$, and we want to calculate the $(m+1)$th eigenvector $\boldsymbol{r}(m+1)$, we modify the initial nonorthogonal value $\bar{\boldsymbol{r}}_0$ by [Eqs. (32) and (35)]

$$\alpha_i = \boldsymbol{r}i^T \boldsymbol{M} \bar{\boldsymbol{r}}_0, \qquad i = 1, \ldots, m \tag{41}$$

$$\boldsymbol{r}_0 = \bar{\boldsymbol{r}}_0 - \sum_{i=1}^{m} \alpha_i \boldsymbol{r}i \tag{42}$$

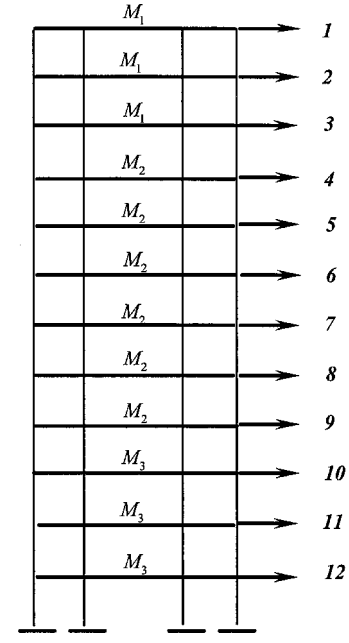
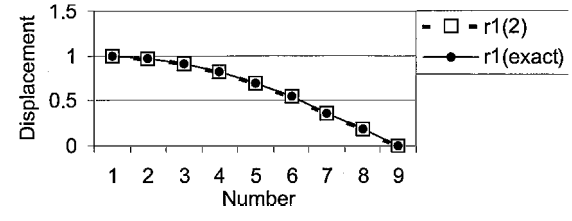
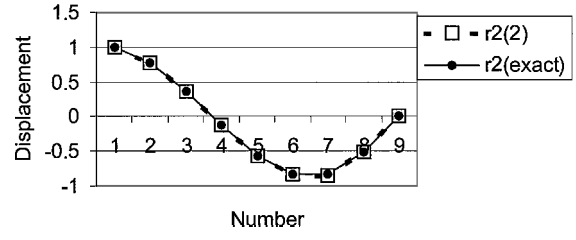


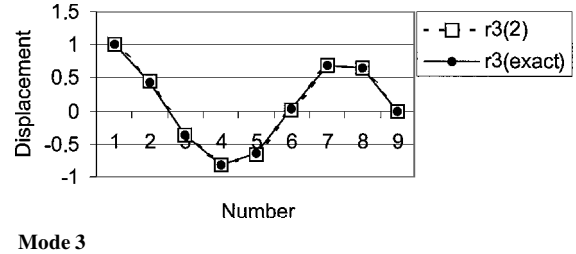
Fig. 3 Frame, 12 stories.



Mode 1



Mode 2



Mode 3

Fig. 4 Stiffness changes in all stories, eight-story frame.

Table 3 Mode shape 1–3 stiffness changes in all stories, 12-story frame

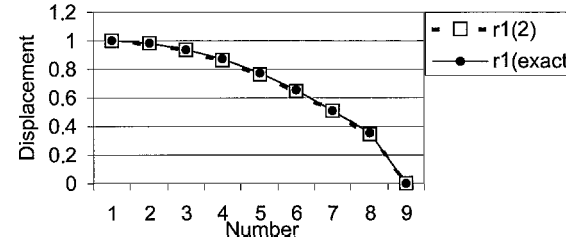
	Value for mode:								
Parameter	1	1	1	2	2	2	3	3	3
Case	$r1(0)$	$r1(2)$	$r1$(exact)	$r2(0)$	$r2(2)$	$r2$(exact)	$r3(0)$	$r3(2)$	$r3$(exact)
Shape	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00
	0.99	0.99	0.99	0.90	0.89	0.89	0.75	0.73	0.73
	0.96	0.96	0.96	0.72	0.69	0.68	0.31	0.26	0.26
	0.93	0.92	0.92	0.46	0.40	0.40	−0.21	−0.28	−0.28
	0.87	0.86	0.86	0.14	0.09	0.09	−0.65	−0.66	−0.67
	0.81	0.79	0.79	−0.21	−0.24	−0.24	−0.84	−0.81	−0.81
	0.72	0.71	0.70	−0.52	−0.53	−0.54	−0.72	−0.66	−0.65
	0.63	0.61	0.60	−0.76	−0.75	−0.75	−0.33	−0.26	−0.24
	0.52	0.49	0.49	−0.88	−0.85	−0.85	0.19	0.25	0.25
	0.40	0.37	0.37	−0.88	−0.83	−0.82	0.63	0.66	0.65
	0.27	0.25	0.25	−0.70	−0.65	−0.65	0.76	0.73	0.72
	0.14	0.13	0.13	−0.39	−0.36	−0.36	0.51	0.47	0.47
λ	0.7	8.4	8.4	5.9	65.7	65.7	15.1	163.4	163.4

Table 4 Mode shape 1–3 stiffness and mass changes in all stories, eight-story frame

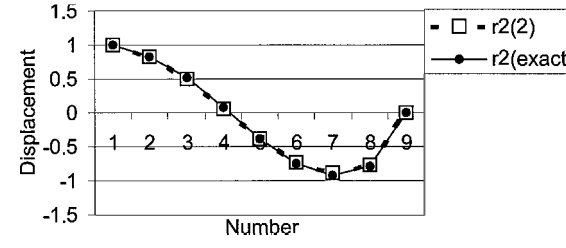
	Value for mode:					
Parameter	1	1	2	2	3	3
Case	$r1(2)$	$r1$(exact)	$r2(2)$	$r2$(exact)	$r3(2)$	$r3$(exact)
Shape	1	1	1	1	1	1
	0.97	0.96	0.68	0.67	0.22	0.14
	0.90	0.89	0.11	0.12	−0.86	−0.84
	0.80	0.78	−0.39	−0.42	−1.12	−1.08
	0.68	0.66	−0.82	−0.83	−0.60	−0.47
	0.52	0.51	−1.02	−0.99	0.48	0.50
	0.34	0.34	−0.88	−0.86	1.27	1.08
	0.17	0.17	−0.52	−0.50	0.90	0.84
λ	11.5	11.6	99.3	99.1	260.2	257.5

Table 5 Mode shape 1–3 stiffness change in the bottom story, eight-story frame

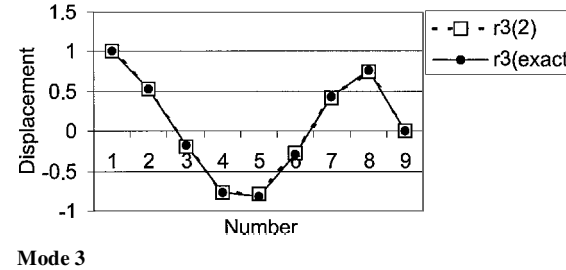
	Value for mode:					
Parameter	1	1	2	2	3	3
Case	$r1(2)$	$r1$(exact)	$r2(2)$	$r2$(exact)	$r3(2)$	$r3$(exact)
Shape	1	1	1	1	1	1
	0.98	0.98	0.82	0.83	0.52	0.53
	0.94	0.94	0.51	0.52	−0.20	−0.18
	0.86	0.87	0.06	0.09	−0.77	−0.77
	0.77	0.77	−0.39	−0.38	−0.79	−0.82
	0.65	0.65	−0.73	−0.74	−0.27	−0.30
	0.51	0.51	−0.88	−0.92	0.41	0.43
	0.35	0.35	−0.77	−0.79	0.74	0.76
λ	1.2	1.2	10.1	10.1	28.0	27.9



Mode 1



Mode 2



Mode 3

Fig. 5 Stiffness change in the bottom story, eight-story frame.

It has been found that using the Gram–Schmidt orthogonalization of Eqs. (41) and (42) for all basis vectors, $\boldsymbol{r}_1, \boldsymbol{r}_2, \boldsymbol{r}_3, \ldots$ (not only for the first basis vector $\boldsymbol{r}_1 = \boldsymbol{r}_0$), significantly improves the accuracy of the results. The procedure presented is very effective, but in some special cases it might be necessary to use mixed basis vectors from more than a single mode (example 4, Sec. IV.D).

2) Calculate the reduced matrices $\boldsymbol{K}_R$ and $\boldsymbol{M}_R$ [Eqs. (13)]

$$\boldsymbol{K}_R = \boldsymbol{r}_B^T \boldsymbol{K} \boldsymbol{r}_B, \qquad \boldsymbol{M}_R = \boldsymbol{r}_B^T \boldsymbol{M} \boldsymbol{r}_B \tag{43}$$

3) Solve the reduced eigenproblem of Eq. (14)

$$\boldsymbol{K}_R \boldsymbol{y} = \lambda \boldsymbol{M}_R \boldsymbol{y} \tag{44}$$

where λ is the first eigenvalue and $\boldsymbol{y}$ is the corresponding eigenvector. The solution at this step is similar to that of Eq. (1), except that a much smaller system is considered.

4) Evaluate the displacement elements of the eigenvector by [Eqs. (9) and (10)]

$$\bar{\boldsymbol{r}}(s) = \boldsymbol{r}_B \boldsymbol{y} \tag{45}$$

$$\boldsymbol{r}(s) = \bar{\boldsymbol{r}}(s)/\mathrm{ref}[\bar{\boldsymbol{r}}(s)] \tag{46}$$

Note that the eigenvalue λ is already given from the solution of Eq. (44).

IV. Numerical Examples

The object of the numerical examples presented in this section is to illustrate typical results achieved by the CA method for different design situations. Some of the examples have been solved by other methods in previous studies. Arbitrary units have been used in all examples.

Small-scale examples are considered for illustrative purposes. The examples demonstrate the accuracy of the results achieved by the method; similar accuracy has been achieved for larger problems. Some of the examples describe common cases such as global and local changes in the design and changes in the stiffness and in

Table 6 Mode shapes 1–4, simply supported beam (eight segments) with a spring

	Value for mode:											
Parameter	1	1	1	2	2	2	3	3	3	4	4	4
Case	$r1(0)$	$r1(2)$	$r1$(exact)	$r2(0)$	$r2(2)$	$r2$(exact)	$r3(0)$	$r3(2)$	$r3$(exact)	$r4(0)$	$r4(2)$	$r4$(exact)
Shape	0.38	0.75	0.79	−0.71	−0.71	−0.71	0.92	0.83	0.82	0.00	0.00	0.00
	0.71	1.00	1.00	−1.00	−1.00	−1.00	0.71	0.11	0.03	−1.00	−1.00	−1.00
	0.92	0.62	0.58	−0.71	−0.71	−0.71	−0.38	−1.00	−1.00	0.00	0.00	0.00
	1.00	0.22	0.21	0.00	0.00	0.00	−1.00	−0.96	−0.95	1.00	1.00	1.00
	0.92	0.62	0.58	0.71	0.71	0.71	−0.38	−1.00	−1.00	0.00	0.00	0.00
	0.71	1.00	1.00	1.00	1.00	1.00	0.71	0.11	0.03	−1.00	−1.00	−1.00
	0.38	0.75	0.79	0.71	0.71	0.71	0.92	0.83	0.82	0.00	0.00	0.00
λ	0.02	0.80	0.78	0.38	0.38	0.38	1.92	5.40	5.39	6.00	6.00	6.00

Table 7 Mode shapes 1–4, simply supported beam (10 segments) with a spring

	Value for mode:											
Parameter	1	1	1	2	2	2	3	3	3	4	4	4
Case	$r1$ (0)	$r1(2)$	$r1$(exact)	$r2(0)$	$r2(2)$	$r2$(exact)	$r3(0)$	$r3(2)$	$r3$(exact)	$r4(0)$	$r4(2)$	$r4$(exact)
Shape	0.31	0.65	0.67	0.62	−0.62	−0.62	0.81	−0.77	−0.85	−1.00	−1.00	−1.00
	0.59	1.00	1.00	1.00	−1.00	−1.00	0.95	−0.48	−0.43	−0.62	−0.62	−0.62
	0.81	0.90	0.84	1.00	−1.00	−1.00	0.31	0.56	0.69	0.62	0.62	0.62
	0.95	0.44	0.39	0.62	−0.62	−0.62	−0.59	1.00	1.00	1.00	1.00	1.00
	1.00	0.12	0.11	0.00	0.00	0.00	−1.00	0.63	0.63	0.00	0.00	0.00
	0.95	0.44	0.39	−0.62	0.62	0.62	−0.59	1.00	1.00	−1.00	−1.00	−1.00
	0.81	0.90	0.84	−1.00	1.00	1.00	0.31	0.56	0.69	−0.62	−0.62	−0.62
	0.59	1.00	1.00	−1.00	1.00	1.00	0.95	−0.48	−0.43	0.62	0.62	0.62
	0.31	0.65	0.67	−0.62	0.62	0.62	0.81	−0.77	−0.85	1.00	1.00	1.00
λ	0.01	0.36	0.35	0.16	0.16	0.16	0.79	2.96	2.90	2.48	2.48	2.48

the mass for various sizes of the structure. Other examples demonstrate the performance of the method in special cases such as switch of modes due to changes in the design and problems with identical eigenvalues. In all examples, approximate results achieved by the CA method with only two basis vectors, $\boldsymbol{ri}(2)$, are compared with the exact displacements, $\boldsymbol{ri}$(exact), for several mode shapes, $i = 1, 2, \ldots$.

A. Example 1: Cantilever Beam

The object of this example, which has been solved by another reanalysis method in a previous study,[16] is to demonstrate the accuracy achieved by the CA method for changes only in the mass matrix. Consider the cantilever beam shown in Fig. 1. The beam is divided into six elements, the initial stiffness of each beam element is given by $EI/L^3 = EI/L^2 = EI/L = 1.0$, and the number of degrees of freedom is 12. The mass of the beam is lumped at the joints, the initial mass at the interior joints is 1.0, and the initial mass at the end of the beam is 0.5. Adding a mass of 2.0 at each of the joints represents the change in the design. Approximate results achieved with only two basis vectors $\boldsymbol{ri}(2)$, the exact displacements $\boldsymbol{ri}$(exact), and the corresponding eigenvalues λ for the first four mode shapes, $i = 1, 2, 3, 4$, are shown in Table 1.

B. Example 2: Frames, 8 and 12 Stories

Consider the 8-story and the 12-story frames shown in Figs. 2 and 3. The mass of the frames is lumped in the girders, with initial values $M_1 = 1.0$, $M_2 = 1.5$, and $M_3 = 2.0$. The girders are assumed to be nondeformable and the initial lateral stiffness of each of the stories is given by $EI/L^3 = 5.0$. To illustrate results for large changes in the lateral stiffness, assume of the modified design that 1) $EI/L^3 = 50.0$ (stories 1–2 from top in the 8-story frame, stories 1–3 in the 12-story frame), 2) $EI/L^3 = 55.0$ (stories 3–6 from top in the 8-story frame, stories 4–9 in the 12-story frame, and 3) $EI/L^3 = 60.0$ (stories 7–8 from top in the 8-story frame, stories 10–12 in the 12-story frame). That is, the stiffness matrix is increased nonuniformly by an order of magnitude. Specifically, the elements of the modified stiffness matrix are about 10–12 times larger then the elements of the initial matrix. Approximate results achieved by the CA method with only two basis vectors $\boldsymbol{ri}(2)$ and the exact displacements $\boldsymbol{ri}$(exact) for the first three mode shapes, $i = 1, 2, 3$, are shown in Tables 2 and 3 and in Fig. 4. The results show that the accuracy of the approximations for both frames is very similar. Note that identical results for the mode shapes would be obtained for the modified stiffness matrices $\boldsymbol{K}$ and $\tau\boldsymbol{K}$, where τ is a scalar multiplier. That is, the accuracy depends on the nature of change in the design and not on the magnitude of change.

To illustrate results for changes in both the stiffness and mass matrices, assume the preceding changes in the stiffness of the eight-story frame and a modified mass of $M = 2.0$ in all stories. The results are summarized in Table 4.

To illustrate results for local design changes, assume a reduction of 50% in the stiffness of the bottom story of the eight-story frame. The modified stiffness for this story is $EI/L^3 = 2.5$. Approximate and exact results for the first three mode shapes are shown in Table 5 and in Fig. 5.

Note that high accuracy is achieved in most cases with only two basis vectors. In cases where better accuracy is required, additional basis vectors could be considered.

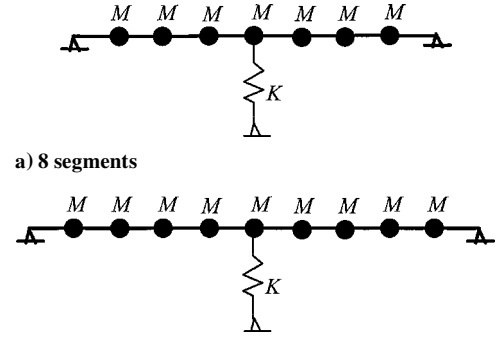
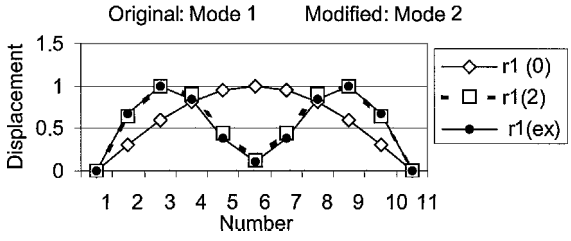


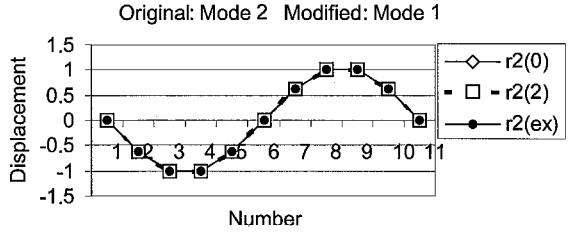
Fig. 6 Simply supported beam with a spring.

C. Example 3: Simply Supported Beam

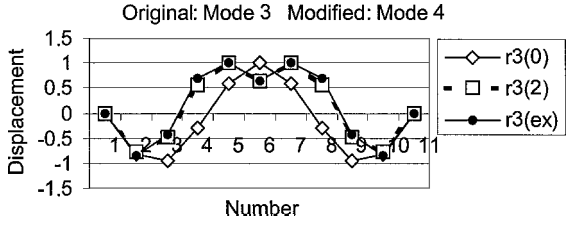
To illustrate results for cases of switch of modes due to changes in the design, consider the two cases of a simply supported beam with a spring in the middle, shown in Fig. 6. The beam is divided



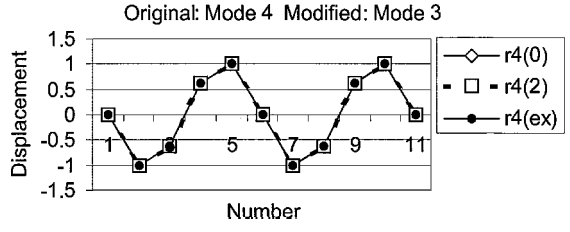
Original mode 1



Original mode 2



Original mode 3



Original mode 4

Fig. 7 Simply supported beam, 10 segments.

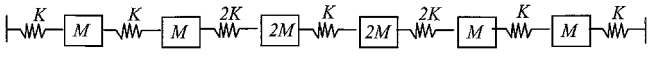
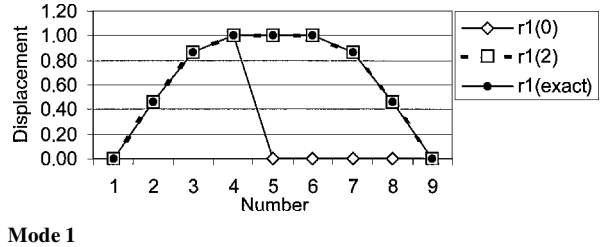
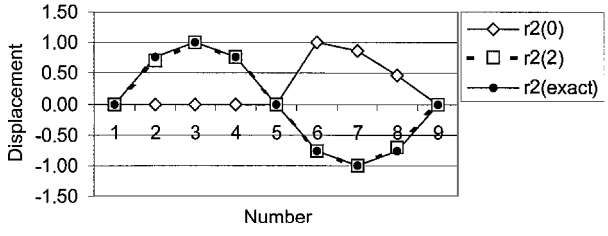


Fig. 8 System of masses and springs.



Mode 1



Mode 2

Fig. 9 System of masses and springs.

Table 8 Mode shapes 1 and 2, system of masses and springs

	Value for mode:					
Parameter	1	1	1	2	2	2
Case	$r1(0)$	$r1(2)$	$r1$(exact)	$r2(0)$	$r2(2)$	$r2$(exact)
Shape	0.46	0.46	0.46	0.00	0.71	0.77
	0.86	0.86	0.86	0.00	1.00	1.00
	1.00	1.00	1.00	0.00	0.76	0.77
	0.00	1.00	1.00	0.00	0.00	0.00
	0.00	1.00	1.00	1.00	−0.76	−0.77
	0.00	0.86	0.86	0.86	−1.00	−1.00
	0.00	0.46	0.46	0.46	−0.71	−0.77
λ	0.139	0.139	0.139	0.139	0.699	0.697

Table 9 Summary of eigenvalues

Design change	Mode	Initial	Modified approximate	Modified exact
	Cantilever beam			
General mass	1	0.0093	0.0025	0.0025
	2	0.3438	0.0990	0.0990
	3	2.5562	0.7743	0.7743
	4	9.2524	2.9352	2.9352
	Frame, 8 stories			
General stiffness	1	1.6	18.0	18.0
	2	12.4	138.5	138.5
	3	31.2	337.4	337.3
General stiffness + mass	1	1.6	11.5	11.6
	2	12.4	99.3	99.1
	3	31.2	260.2	257.5
Bottom story stiffness	1	1.6	1.2	1.2
	2	12.4	10.1	10.1
	3	31.2	28.0	27.9
	Frame, 12 stories			
General stiffness	1	0.7	8.4	8.4
	2	5.9	65.7	65.7
	3	15.1	163.4	163.4
	Beam, 8 segments			
Adding a spring	1	0.02	0.80	0.78
	2	0.38	0.38	0.38
	3	1.92	5.40	5.39
	Beam, 10 segments			
Adding a spring	1	0.01	0.36	0.35
	2	0.16	0.16	0.16
	3	0.79	2.96	2.90
	4	2.48	2.48	2.48
	System of masses and springs			
Adding a spring	1	0.14	0.14	0.14
	2	0.14	0.70	0.70

into 8 (16 degrees of freedom; Fig. 6a) and 10 elements (20 degrees of freedom; Fig. 6b). The initial design in both cases is given by 1) lumped masses, $M = 1.0$; 2) the stiffness of each beam element, $EI/L^3 = EI/L^2 = EI/L = 1.0$; and 3) the stiffness of the spring, $K = 0$ (there is no spring).

The modified design in both cases is represented by adding a spring with $K = 12$ in the middle of the span. The initial displacements, results achieved by the CA method with only two basis vectors, and the final exact displacements for the first four mode shapes are shown in Tables 6 and 7 and in Fig 7. Note that the change in the design causes significant changes in the mode shapes and in the eigenvalues. Moreover, there is a switch in the modes as follows. In both cases, the original mode 1 becomes the modified mode 2, and the original mode 2 becomes the modified mode 1. In addition, for the 10-segments beam, the original mode 3 becomes the modified mode 4, and the original mode 4 becomes the modified mode 3. It is observed that high accuracy has been achieved by the CA method with only two basis vectors.

D. Example 4: System of Masses and Springs

To illustrate results for problems with identical eigenvalues and to compare results achieved by other methods, consider the system

shown in Fig. 8. This example has been solved by another reanalysis method in a previous study.[16] The initial lumped masses are given by $M = 1$, the initial stiffness is given by $K = 1$, and $K = 0$ for the middle spring (there is no spring). That is, the initial system consists of two independent subsystems. The modified design is represented by adding the middle spring with $K = 1$. In this case, it is necessary to consider basis vectors of the two subsystems. With only two basis vectors (the first mode of each of the two subsystems), the exact modified shape of mode 1 is obtained. For mode 2, an approximate solution is obtained with the usual two basis vectors. The initial displacements, results achieved by the CA method with only two basis vectors, and the exact displacements for the first two mode shapes are shown in Table 8 and in Fig. 9. A summary of the various eigenvalues calculated in the examples is given in Table 9.

V. Conclusions

In this study, the CA method has been developed for vibration reanalysis problems. Different types of structures and various types of changes in the design may be considered. The calculations are based on results of a single exact analysis, and the computational effort is usually much smaller than the effort needed for complete vibration analysis of the modified design.

The efficiency of the calculations and the accuracy of the results can be controlled by the amount of information considered. That is, high accuracy can be achieved by considering higher-order terms. The solution steps are straightforward, the method can be readily used with a general finite element system, and calculation of derivatives is not required. This makes the method more attractive in various applications where derivatives are not available.

The examples demonstrate the solution methodology and typical numerical results. Small-scale examples have been selected for this purpose. Solutions of various examples show that similar results are obtained for larger structures. Accurate results are achieved efficiently for very large changes in the design. In the examples solved, high accuracy is achieved for various cases with only two basis vectors. Better accuracy could be achieved, when necessary, by considering additional basis vectors. In particular cases, it might be necessary to consider mixed basis vectors from more than a single mode.

Further research is needed to study the accuracy of the results for various structures where numerical problems may arise.

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